PHYSICAL REVIEW D, VOLUME 62, 064023

Moduli spaces for four- and five-dimensional black holes

J. Gutowski
Center for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, England

G. Papadopoulos
Department of Mathematics, King's College London, Strand, London WC2R 2LS, England

(Received 21 March 2000; published 24 August 2000)

We propose a universal expression for the moduli metric of a class of four- and five-dimensional black holes which preserve at least four supersymmetries. These include the black holes that are associated with various intersecting branes in ten and eleven dimensions, the electrically charged black holes of $N=2$, $D=5$ and $N=2$, $D=4$ supergravities with any number of vector multiplets, and dyonic black holes of $N=2$, $D=4$ supergravity. The moduli metric of electrically charged $N=2$, $D=4$ black holes coupled to any number of vector multiplets is explicitly computed. We also investigate the superconformal symmetries of the black hole moduli spaces for small black hole separations.

PACS number(s): 04.65.+e

I. INTRODUCTION

In the past few years there has been much interest in investigating the geometry of the moduli spaces of various supersymmetric black hole solutions of supergravity theories following some earlier work in [1–3]. Supersymmetric black hole solutions are thought as the solitons of supergravity and so provide a macroscopic description of the solitons of strings and M theory. As such one can investigate their moduli spaces in analogy with similar investigations of the moduli spaces of Bogomol'nyi-Prasad-Sommerfield (BPS) monopoles in the context of Yang-Mills theory. However unlike the case of BPS monopoles, the geometry of the moduli space of various black holes is related to that of the target space of supersymmetric sigma models in one dimension with *nonvanishing torsion* [4]; for more recent work on the geometry of one-dimensional sigma models see [5]. This has been first established in [6] for a class of $D=5$ black holes that preserve 1/4 of the maximal supersymmetry and later extended for the electric black holes of $D=5$, $N=2$ supergravity coupled to the graviphoton [7]. In the former case, the geometry of black hole moduli space is a strong hyper-Kähler manifold with torsion (HKT) while in the latter is weak HKT [8]. Later it was found that the moduli space of electrically charged black holes of $N=2$, $D=5$ supergravity with any number of vector multiplets is again weak HKT [9]. More recently, the moduli space of (four-dimensional) Reissner-Nordström black holes was investigated [10] and again it was confirmed that its geometry is related to that of a class of one-dimensional sigma models which in addition to some bosonic multiplets also contain fermionic ones [4]. A common characteristic of all the above cases is that the geometry on the moduli space of these black holes is determined by a scalar function, a ''moduli potential.'' The authors of [7,10] also investigated the symmetries of the moduli spaces of the $N=2$, $D=5$ graviphoton and the Reissner-Nordström black holes for small black hole separations and they found that they exhibit a $D(2,1;0)$ superconformal symmetry.

In this paper, we propose a moduli metric for a class of black holes in four and five dimensions that preserve at least four supersymmetry charges of the underlying supergravity theory. Typically, we consider black holes of maximal supergravities or black holes of $N=2$ supergravities in four and five dimensions. To be specific, if the metric of supersymmetric black holes is given by

$$ds^2=-A^2dt^2+B^2d\mathbf{x}d\mathbf{x}, \tag{1.1}$$

then the moduli potential μ is

$$\mu=\int d^{D-1}xA^{-2}B^2, \tag{1.2}$$

where $D=5$ or $D=4$. The integration is over the spatial directions of the black hole with respect to the Euclidean metric. It is assumed that the solution is perturbed within the appropriate supergravity theory and the only moduli parameters are the positions of the black holes. The above choice of the moduli potential is independent from the choice of frame of the associated supergravity theory. One novel property of our expression for the moduli potential is that it includes all the examples of known black hole moduli potentials that have been computed so far. We also explicitly compute the moduli metric of the electrically charged black hole solutions of $N=2$, $D=4$ supergravity [11] in Sec. VIII and we find that it is again given by Eq. (1.2). In addition, we verify that the effective action associated with the moduli potential (1.2) of a certain class of black holes exhibits the expected super-conformal symmetries for small black hole separations. The metric and torsion on the black hole moduli space associated with the moduli potential (1.2) are given in Secs. VI and VII.

We first apply our formula to four- and five-dimensional black holes that can be constructed by reducing intersecting brane configurations from ten and eleven dimensions. We show that the moduli potential is invariant under the T and S dualities of type II strings and independent from the choice of frame of the supergravity theories. Moreover, Eq. (1.2) can be partially motivated by the universality of the ratio of the conformal factor of the transverse directions of the branes modulo that of the worldvolume directions. Since the moduli space of a class of electrically charged black holes has been explicitly computed, if one assumes that the moduli metric is S- and T-duality invariant, then one can establish Eq. (1.2). Another application of the moduli potential (1.2) is in establishing the moduli metric of intersecting brane configurations as will be explained in section three.

We next apply our formula (1.2) to a class of four- and five-dimensional black holes associated with $N=2$, $D=5$ supergravity with any number of vector multiplets. The moduli potential of the electrically charged black holes of this theory has been explicitly computed and agrees with Eq. (1.2). This includes the case of five-dimensional black holes coupled to the graviphoton and investigated in [7]. In four dimensions our formula agrees with the moduli potential computed from the moduli metric of (four-dimensional) Reissner-Nordström black holes. Next we apply Eq. (1.2) to give the moduli potential of the dyonic four-dimensional black holes that arise from the reduction of the string solutions of $N=2$, $D=5$ supergravity superposed with a pp wave and the electrically charged $N=2$, $D=5$ black holes in the background of a Kaluza-Klein (KK) monopole. We also verify with an explicit computation that the moduli potential of the four-dimensional electrically charged black holes of $N=2$, $D=4$ supergravity is again given by Eq. (1.2). We remark that the above mentioned dyonic black holes are dual to these electric ones. Finally, we investigate the supersymmetric and superconformal properties of the effective actions of all the above black holes. We find that for the class of such black holes which have regular horizons the effective action exhibits $D(2,1;0)$ superconformal symmetry at small black hole separations.

This paper has been organized as follows: In Sec. II, we describe the moduli potential of four- and five-dimensional black holes that arise from brane intersections in ten and eleven dimensions. In Sec. III, we provide evidence in support of Eq. (1.2) using duality. In Sec. IV, we give the four-dimensional dyonic black hole solutions which are reductions of the string solutions superposed with a pp wave and the electrically charged solutions superposed with a KK monopole of $N=2$, $D=5$ supergravity. In Sec. V, we apply our formula to give the moduli potential of all the above black holes and as an example we present the moduli potential of black holes associated with the STU model. In Sec. VI, we construct the effective theory of five-dimensional black holes and examine its superconformal properties. In Sec. VII, we construct the effective theory of four-dimensional black holes and examine its superconformal properties. In Sec. VIII, we compute the moduli metric of the electrically charged four-dimensional black holes of $N=2$ supergravity coupled to any number of vector multiplets and in Sec. IX we give our conclusions.

II. BLACK HOLES IN FOUR AND FIVE DIMENSIONS

A large class of four- and five-dimensional black holes[1] can be constructed by appropriately reducing intersecting brane configurations of strings and M theory [12]. This has been widely explored in the literature [12–15]. These in particular include the black holes that have been used to perform a microscopic computation of the Bekenstein-Hawking entropy in [16,17]. It has been observed in [14] that the metric of such black hole solutions can be expressed as

$$ds^2=-\lambda^{D-3}dt^2+\lambda^{-1}ds^2(\mathbb{E}^{D-1}), \tag{2.1}$$

where

$$\lambda=(\Pi_{I=1}^{n}H_I)^{-1/(D-2)}, \tag{2.2}$$

and

$$\Pi_{I=1}^{n}H_I=H_1\ldots H_n. \tag{2.3}$$

The functions H_I are harmonic in $\mathbb{E}^{D-1}$, i.e.,

$$H_I=h_I+\sum_A \frac{\lambda_{IA}}{|\mathbf{x}-\mathbf{y}_{IA}|^{D-3}}. \tag{2.4}$$

The constants $\{h_I;I=1,\ldots,n\}$ can be related to the asymptotic values of the various scalars of the supergravity theory, $\{\lambda_{IA};A=1,\ldots,N_I;I=1,\ldots,n\}$ are the black hole charges and $\{\mathbf{y}_{IA};A=1,\ldots,N_I;I=1,\ldots,n\}$ are the black hole positions. For five-dimensional black holes ($D=5$) $n\leqslant 3$ and for four-dimensional black holes ($D=4$) $n\leqslant 4$. A subclass of such black holes are those for which the positions of the harmonic functions coincide, i.e.,

$$\mathbf{y}_{IA}=\mathbf{y}_{JA} \tag{2.5}$$

for $I\neq J$. If in addition $n=3$ for $D=5$ or $n=4$ for $D=4$, then these black holes have regular horizons.

The black holes that are described by the metric (2.1) carry electric or magnetic or both electric and magnetic charges. The origin of these charges can be traced in their interpretation as intersecting branes in ten or eleven dimensions. In broad terms if the black hole is associated with M-2-branes and pp waves, then it is electrically charged but if it is associated with M-5-branes and KK monopoles, then it is magnetically charged. There are also dyonic black holes that are associated with both electric and magnetic branes. The Maxwell fields that the above black holes couple to are either KK vectors due to the reduction or they are associated to the various brane field strengths in ten and eleven dimensions. In some cases, it is possible to take linear combinations of the Maxwell fields such that it can appear that a black hole couples to fewer Maxwell fields than it may be expected. This mostly arises when we set the various harmonic functions that the black holes depend on to be equal.

[1]We use the term black holes to characterize all static solutions of supergravity which are asymptotically flat. In particular, we do not require for the solutions to have an event horizon.

Since, we shall not use the explicit expression of the Maxwell fields of the solutions in what follows we shall neglect them.

Using the expression for the moduli potential proposed in the Introduction, we find that

$$\mu=\int d^{D-1}x\ \lambda^{2-D} \tag{2.6}$$

or equivalently

$$\mu=\int d^{D-1}x\ \Pi_{I=1}^{n}H_I. \tag{2.7}$$

In the five-dimensional case $(D=5)$, for $n=1$ the moduli space is flat, for $n=2$ the moduli space is strong HKT and for $n=3$ the moduli space is weak HKT. Such solutions preserve 1/2, 1/4 and 1/8 of supersymmetry, respectively. Moreover they are associated with configurations in strings and M theory that involve one brane, two branes and three branes, respectively. The general case is that with three harmonic functions since all the others can be derived from it by setting one or more of the harmonic functions to be constant. Another simplification of the $n=3$ case is to set all the harmonic functions to be equal: i.e.,

$$H=H_1=H_2=H_3. \tag{2.8}$$

In that case, the moduli potential becomes

$$\mu=\int d^4x\ H^3. \tag{2.9}$$

This moduli potential is the same as that of the black holes of $N=2$ supergravity coupled to the graviphoton and derived in [7]. In fact, the graviphoton black hole can be constructed by reducing the M-brane configuration of three intersecting M-2-branes on a 0-brane with all three harmonic functions associated with each M-2-brane set to be equal [12]. In this case, the moduli potential (2.9) has been verified by an explicit calculation. For the rest of the cases, we shall provide an argument in the next section.

The superconformal properties of the moduli space (2.9) for small black hole separation are the same as those of graviphoton black holes investigated in [7]. A more general case arises whenever we choose the positions of the harmonic functions to be the same but the asymptotic values of the scalars and the charges to be different: i.e.,

$$H_I=h_I+\sum_A \frac{\lambda_{IA}}{|\mathbf{x}-\mathbf{y}_A|^{D-3}}. \tag{2.10}$$

The superconformal properties of these black holes will be investigated in Sec. VI.

In the four-dimensional case $(D=4)$, for $n=1$ the moduli space is flat and for $n=4$ the geometry on the moduli space generalizes that on the moduli space of the Reissner-Nordström black holes. The rest of the cases are new. For $n=1$ the solutions preserve 1/2, for $n=2$ the solutions preserve 1/4, for $n=3$ and $n=4$ the solutions preserve 1/8 of the maximal supersymmetry, respectively. As in the five-dimensional case, the most general case arises whenever $n=4$ since all the other cases can be derived by setting one or more harmonic functions to be constant. Another simplification of the $n=4$ case is to set all the harmonic functions to be equal: i.e.,

$$H=H_1=H_2=H_3=H_4. \tag{2.11}$$

In that case, the moduli potential becomes

$$\mu=\int d^3xH^4. \tag{2.12}$$

The geometry on the moduli space then is that of the Reissner-Nordström black holes [10]. So the superconformal properties of the effective theory for small black hole separation are the same as those of the Reissner-Nordström black holes. A more general case arises whenever we choose the positions of the harmonic functions to be the same but the asymptotic values of the scalars and the charges to be different as in Eq. (2.10). The superconformal properties of these black holes will be investigated in Sec. VII.

III. MODULI SPACES FROM INTERSECTING BRANES

It is well known that all of the supersymmetric black holes in four and five dimensions can be constructed by reducing intersecting M-brane configurations possibly superposed with a pp wave or a KK monopole. The two latter configurations reduce to a D-0-brane and a D-6-brane upon compactification to ten dimensions on S^1, respectively [18]. So one may expect to understand the expression of the moduli potential by investigating the brane solutions in eleven and ten dimensions. We shall argue that indeed some of the features of the proposed moduli potential of the four- and five-dimensional black holes have their origin in the form of the brane solutions in ten and eleven dimensions. But there are also some puzzles.

The first observation toward this concerns the ratio of the components of the metric of all brane solutions in ten and eleven dimensions. To be specific recall that the spacetime metrics of the various branes are as follows: The metric of the M-2-brane [19] and the M-5-brane [20] are

$$\begin{aligned} ds^2&=H^{1/3}[H^{-1}ds^2(\mathbb{E}^{(1,2)})+ds^2(\mathbb{E}^8)],\\ ds^2&=H^{2/3}[H^{-1}ds^2(\mathbb{E}^{(1,5)})+ds^2(\mathbb{E}^5)], \end{aligned} \tag{3.1}$$

respectively, the metrics of the D-p branes [21,22] are

$$ds^2=H^{-1/2}ds^2(\mathbb{E}^{(1,p)})+H^{1/2}ds^2(\mathbb{E}^{9-p}), \tag{3.2}$$

the metric of the fundamental string [23] is

$$ds^2=H^{-1}ds^2(\mathbb{E}^{(1,1)})+ds^2(\mathbb{E}^8), \tag{3.3}$$

and the metric of the NS-5-brane [24] is

$$ds^2=ds^2(\mathbb{E}^{(1,5)})+Hds^2(\mathbb{E}^4), \tag{3.4}$$

where H is a harmonic function of the transverse directions in each case. A common characteristic of all these solutions is that the ratio B^2A^{-2} of the conformal factor of the transverse directions modulo that of the world volume directions is equal to H,

$$B^2A^{-2}=H. \tag{3.5}$$

In particular this implies that this ratio is invariant under the various T and S dualities that relate the M theory and type II string theories. It also follows from the above observation and the harmonic function rule [25] that for all the intersecting brane configurations the ratio of the conformal factor of the *overall transverse* directions modulo that of the *common intersection* is universal and depends only on the number n of the branes involved in the intersection. In particular, we find that

$$B^2A^{-2}=H_1H_2\cdots H_n. \tag{3.6}$$

Incidentally, this ratio is the same as that of the spatial modulo the timelike components of the metric of the associated black holes.

One expects that the moduli space of four- and five-dimensional black holes, that arise from appropriately reducing the above brane solutions is flat. This is because the associated black holes preserve 1/2 of the maximal supersymmetry and so the effective action has sixteen supersymmetries. Such a high number of supersymmetries render the sigma model target space flat.[2] In such case, one can choose for the black hole moduli potential

$$\mu=\int d^{D-1}xH. \tag{3.7}$$

Now if two or more branes are involved in the configuration, it is natural to take the moduli potential to depend on the product of harmonic functions. This is because all harmonic functions enter in a symmetric way in the black hole metric[3] and that if one of them is set to one, say $H_k=1$, $1\leqslant k\leqslant n$, the expression for the moduli potential should remain symmetric in the rest of the harmonic functions. Of course there are many other symmetric polynomials of the harmonic functions which can be added in the expression for the potential. But all of them have degree lower than that of the product. After setting one or more harmonic functions to one, we get a moduli metric which would be scaled by a conformal factor. In particular, we shall find a scaled version of the potential (3.7) that is not expected.

Another argument in support of Eq. (1.2) can be established using duality. As we have mentioned the expression for the moduli potential is T and S dualities invariant. Now if we assume that black holes that are related by T and S dualities should have the same moduli space, then the moduli potential (1.2) can be derived in the five-dimensional case. This is because the moduli potential for the graviphoton black hole agrees with Eq. (1.2) and that this black hole is a reduction from the M-theory configuration of three M-2-branes intersecting on a 0-brane. The M-2-brane configuration is then related to the rest of intersecting branes of strings and M theory via T and S dualities which give the rest of five-dimensional black holes.

It is worth mentioning that our expression for the moduli potential (1.2) can apply to intersecting brane configurations. In this case, the effective theory may not be one-dimensional. Typically, the dimension of the effective theory is that of the common intersection. Moreover, the effective theory may contain apart from scalars other fields like vectors and tensors. However, we argue that the part of the effective theory which describes the dynamics of the scalars that are associated with the *overall position* of the configuration in spacetime, when reduced to one-dimension, coincides with the effective theory of the black hole that can be constructed from the configuration. To give an example, let us consider the moduli of the solution of eleven-dimensional supergravity with metric

$$\begin{aligned}ds^2={}&H_1^{1/3}H_2^{1/3}H_3^{1/3}[-H_1^{-1}H_2^{-1}H_3^{-1}ds^2(\mathbb{E})+H_1^{-1}ds^2(\mathbb{E}^2)\\&+H_2^{-1}ds^2(\mathbb{E}^2)+H_3^{-1}ds^2(\mathbb{E}^2)+ds^2(\mathbb{E}^4)],\end{aligned} \tag{3.8}$$

which has the interpretation of three M-2-branes intersecting on a 0-brane, where H_1,H_2,H_3 are harmonic functions on $\mathbb{E}^4$. The effective theory of the transverse scalars this configuration is determined by the moduli potential

$$\mu=\int d^4xH_1H_2H_3, \tag{3.9}$$

which is the ratio of the component of the metric along $\mathbb{E}^4$ (overall transverse) to the component of the metric along $\mathbb{E}$ (common intersection). The above moduli potential is of course the moduli potential of the five-dimensional black hole associated with the above configuration. A similar analysis can be done for other such configurations.

Despite these there are some puzzles. One involves the M-theory configuration of three M-5-branes pairwise intersecting on a 3-brane and all together at a string [12]. Reducing this solution to five dimensions, we find a string solution which is in the same universality class as the string solutions of the $N=2$, $D=5$ supergravity. One might expect that the effective theory of such strings using the supersymmetry projectors of the M-branes to be a (4,0)-supersymmetric two-dimensional sigma model with strong HKT geometry. This would imply that the moduli space has dimension $4N$ and that the torsion is a closed three form. Since there are three transverse scalars for each black hole, the moduli space has in fact dimension $3N$ and the torsion is not a closed form. The former point can be explained by arguing that there are additional moduli for these black holes. Indeed in string theory apart from the transverse scalars, intersecting brane configurations have additional scalar, vector and or even ten-

[2]It also follows from the reduction of the effective theories of branes to lower dimensions neglecting possible non-Abelian interactions and collecting the terms quadratic in the velocities.

[3]There are completely symmetric brane intersections that give rise to four- and five-dimensional black holes. For these the metric, the scalars and the Maxwell fields are all symmetric.

sor moduli. The additional scalars may be due to, e.g., D-brane type of counting; for an example see [26]. All these can be reduced to five dimensions giving a bigger moduli space from the one we are investigating. The latter point may also be resolved by observing that the intersection is chiral. In such a case the torsion can be modified by adding a Chern-Simons type of term to cancel the anomaly which renders the modified torsion to be a nonclosed form. However, we have not been able to establish the details of the above arguments. One encounters similar puzzles with even purely electric solutions as it has already been mentioned in [6]. As another one consider the seven-dimensional black hole which can be found by reducing the intersection of two M-2-branes on a 0-brane configuration of M theory. One can easily see that in this case there is missing moduli. However upon reducing the solution further to five-dimensions, the moduli space becomes strong HKT as expected. This appears to be a rather common phenomenon. Upon reducing the solutions to an appropriate dimension, the geometry of the associated black hole moduli space can be understood in terms of that of the target space of a supersymmetric sigma model.

IV. VERY SPECIAL FOUR-DIMENSIONAL BLACK HOLES

A large class of black holes in four dimensions can be constructed by reducing either the string solution of the $N=2$ supergravity superposed with a *pp* wave or the very special electrically charged black holes of the $N=2$, $D=5$ supergravity in a KK-monopole background. To describe these solutions, we first review some aspects of $N=2$, $D=5$ supergravity. The bosonic part of the action of five-dimensional $N=2$ supergravity with k vector multiplets is associated to a hypersurface N of $\mathbb{R}^k$ defined by the equation

$$V(X)\equiv\frac{1}{6}C_{IJK}X^IX^JX^K=1, \tag{4.1}$$

where $\{X^I;I=1,...,k\}$ are standard coordinates on $\mathbb{R}^k$ and C_{IJK} are constants. In the case of a model arising from a Calabi-Yau compactification of M theory, the constants C_{IJK} are the topological intersection numbers of the compact manifold. Next we set

$$Q_{IJ}\equiv-\frac{1}{2}\frac{\partial}{\partial X^I}\frac{\partial}{\partial X^J}\log V|_{V=1}=\frac{9}{2}X_IX_J-\frac{1}{2}C_{IJK}X^K,$$
$$h_{ab}=Q_{IJ}\frac{\partial X^I}{\partial\phi^a}\frac{\partial X^J}{\partial\phi^b}\bigg|_{V=1}, \tag{4.2}$$

where $\{\phi^a;i=1,...,k-1\}$ are local coordinates of N, h is interpreted as a metric on N and

$$X_I=\frac{1}{6}C_{IJK}X^JX^K \tag{4.3}$$

are the dual coordinates to X^I. Note that the hypersurface equation $V=1$ can also be rewritten as $X^IX_I=1$. Then, the bosonic part of the associated supergravity action [27,28] with vector potentials A^I and scalars ϕ^a is

$$S=\int d^5x\sqrt{-g}\left[R+\frac{1}{2}Q_{IJ}F^I{}_{\mu\nu}F^{J\mu\nu}+h_{ab}\partial_\mu\phi^a\partial^\mu\phi^b\right]-\frac{1}{24}e^{\mu\nu\rho\sigma\tau}C_{IJK}F^I{}_{\mu\nu}F^J{}_{\rho\sigma}A^K{}_\tau, \tag{4.4}$$

where $F^I=dA^I, I,J,K=1,\dots,k$ are the 2-form Maxwell field strengths, μ, ν, ρ, $\sigma=0,\dots,4$, and g is the metric of the five-dimensional spacetime; we have used the same symbol ϕ^a to denote both the coordinates of N and the various scalar fields of the theory.

The field equations of the above Lagrangian obtained from varying the scalars ϕ^a, the spacetime metric g, and the vector potentials A^I are

$$\sqrt{-g}\partial_aQ_{IJ}\left[\frac{1}{2}F^I{}_{\mu\nu}F^{J\mu\nu}+\partial_\mu X^I\partial^\mu X^J\right]-2\partial_\mu(\sqrt{-g}Q_{IJ}\partial^\mu X^I)\partial_aX^K=0, \tag{4.5}$$

$$\sqrt{-g}(G_{\mu\nu}+Q_{IJ}F^I{}_{\mu\rho}F^J{}_\nu{}^\rho+Q_{IJ}\partial_\mu X^I\partial_\nu X^J)-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\left[\frac{1}{2}Q_{IJ}F^I{}_{\rho\sigma}F^{J\rho\sigma}+Q_{IJ}\partial_\rho X^I\partial^\rho X^J\right]=0, \tag{4.6}$$

and

$$-2\partial_\mu[\sqrt{-g}Q_{IJ}F^{J\mu\nu}]-\frac{1}{8}e^{\nu\rho\sigma\mu\tau}C_{IJK}F^J{}_{\rho\sigma}F^K{}_{\mu\tau}=0, \tag{4.7}$$

respectively. The electrically charged black holes have been found in [29]. The electrically charged black hole solutions in the background of a KK monopole are

$$ds^2=-e^{-4U}dt^2+e^{2U}[H_0^{-1}(d\tau+\omega)^2+H_0d\mathbf{x}^2],$$
$$A_0^I=e^{-2U}X^I, \tag{4.8}$$

$$e^{2U}X_I=\frac{1}{3}H_I,$$

where

$$H_I=h_I+\sum_{A=1}^{N_t}\frac{\lambda_{IA}}{|\mathbf{x}-\mathbf{y}_{IA}|},$$
$$H_0=h_0+\sum_{A=1}^{N_0}\frac{\lambda_{0A}}{|\mathbf{x}-\mathbf{y}_{0A}|} \tag{4.9}$$

are harmonic functions on $\mathbb{E}^3$. The positions of the black holes are labeled by the coordinates $\{(\mathbf{y}_{0A},\mathbf{y}_{IA});\ A=1,\dots,N_J;\ I=1,\dots,k\}$. The constants $\{(h_0,h_I);\ I=1,\dots,k\}$ are the values of the scalar fields at spatial infinity and $\{(\lambda_{0A},\lambda_{IA});\ A=1,\dots,N_I;\ I=1,\dots,k\}$ are the charges of the black holes. Viewing e^U as an additional scalar, the last equation in Eq. (4.8) gives the k independent scalars $\{e^U,\phi^a\}$ in terms of the k harmonic functions $\{H_I\}$. A reduction of this solution to four dimensions along the compact direction τ leads to four-dimensional black holes with metric (in the Einstein frame)

$$ds^2=-e^{-3U}H_0^{-1/2}dt^2+e^{3U}H_0^{1/2}d\mathbf{x}^2. \tag{4.10}$$

These black holes, apart from the electric charges associated with those in five dimensions, also have a magnetic charge with respect to the KK vector of the reduction.

The other class of four-dimensional black holes can be obtained by using the string solution of $N=2$, $D=5$ supergravity of [30]. Superposing this solutions with a *pp* wave, we have

$$\begin{aligned} &ds^2=e^{-U}(dudv+H^0du^2)+e^{2U}d\mathbf{x}^2,\\ &F^I_{mn}=-\epsilon_{mnp}\partial_pH^I,\\ &e^UX^I=H^I, \end{aligned} \tag{4.11}$$

where u, v are light-come coordinates and

$$\begin{aligned} H^I&=h^I+\sum_{A=1}^{N_I}\frac{\lambda^I_A}{|\mathbf{x}-\mathbf{y}_{IA}|},\\ H^0&=h^0+\sum_{A=1}^{N_0}\frac{\lambda^0_A}{|\mathbf{x}-\mathbf{y}_{0A}|}. \end{aligned} \tag{4.12}$$

The positions of the black holes are labeled by the coordinates $\{(\mathbf{y}_{0A},\mathbf{y}_{IA});\ A=1,\dots,N_I;\ I=1,\dots,k\}$ as in the previous case. The constants $\{(h^0,h^I);\ I=1,\dots,k\}$ are the values of the scalar fields at spatial infinity and $\{(\lambda^0_A,\lambda^I_A);\ A=1,\dots,N_I;\ I=1,\dots,k\}$ are the charges of the black holes. Reducing this solution to four-dimensions along the direction of the wave leads to a four-dimensional black hole with metric in the Einstein frame (see also [11])

$$ds^2=-H_0^{-1/2}e^{-(3/2)U}dt^2+H_0^{1/2}e^{(3/2)U}d\mathbf{x}^2. \tag{4.13}$$

These black holes carry magnetic charges which correspond to the charges of the five-dimensional string. They also carry an electric charge which is related to the momentum of the *pp* wave.

Many other dyonic black hole solutions of four-dimensional supergravity theories are known. Some of them have been found by investigating the solutions of supergravity theories associated with the heterotic string [31]. The relation of these black holes to brane configurations of the heterotic string have also been explored [32].

To express the black hole solutions explicitly in terms of the harmonic functions, one has to solve the stabilization equations (see [33]). From here on, we shall assume that solutions to these equations exist for the models we are considering. We also remark that a special subclass of solutions are those for which the positions of the harmonic functions are the same $\mathbf{y}_{0A}=\mathbf{y}_{IA}=\mathbf{y}_{JA}$ for $I\neq J$. These black holes are of interest because they exhibit regular horizons. It is straightforward to see this by extending the arguments of [34,30] which have followed earlier work in [35–37]. In both cases the near horizon geometry is $AdS_2\times S^2$.

V. THE MODULI POTENTIAL OF $N=2$ BLACK HOLES

The universal formula (1.2) for the moduli potential is in agreement with the explicitly computed moduli potential of the electrically charged black holes of $N=2$, $D=5$ supergravity with any number of vector multiplets [9]. Moreover, Eq. (1.2) is also in agreement with the explicitly computed moduli potential of the Reissner-Nordström black hole in [10].

Applying Eq. (1.2), we find that the moduli potential for the black holes (4.10) is

$$\mu_1=\int d^3xH_0e^{6U}, \tag{5.1}$$

while for the black holes (4.13) is

$$\mu_2=\int d^3xH^0e^{3U}. \tag{5.2}$$

If we again assume that the moduli metric is invariant under duality, then in both cases the moduli potentials can be derived from that which we compute in Sec. VIII. This is because, as we have mentioned, the above dyonic black holes are dual to the electrically charged ones of $N=2$, $D=4$ supergravity.

To give some examples, we have to consider models for which the stabilization equations have an explicit solution. Such a model is the so called STU model [38]; for other models see [39,40]. For this, the only nonvanishing component of C_{IJK} is C_{123}. For the back holes (4.10), one can find that

$$e^{6U}=H_1H_2H_3. \tag{5.3}$$

Therefore, the moduli potential of the associated four-dimensional black holes is

$$\mu_1=\int d^3xH_0H_1H_2H_3. \tag{5.4}$$

Similarly for the Eq. (4.13) black holes, we find that

$$e^{3U}=H^1H^2H^3 \tag{5.5}$$

and, consequently, the moduli potential is

$$\mu_2=\int d^3xH^0H^1H^2H^3. \tag{5.6}$$

It is apparent that the moduli potential in both the above cases is the same. So the geometry on the moduli space of Eq. (4.10) black holes is the same as that on the moduli space of Eq. (4.13) black holes. Moreover, it coincides with the moduli potential of four-dimensional black holes which are associated with intersecting branes and have four harmonic functions in Secs. II and III. As for the intersecting branes black holes, we can set $H=H^0=H^1=H^2=H^3$ or $H=H_0=H_1=H_2=H_3$. Then the moduli potentials μ_1 and μ_2 reduce to that of Reissner-Nordström black hole computed in [10]. A more general case is to take the positions of the harmonic functions of Eq. (4.10) black holes to be the same but allow the charges λ_{IA} and the asymptotic values of the scalars to be different, i.e., $\mathbf{y}_A=\mathbf{y}_{0A}=\mathbf{y}_{IA}$ for $I=1,\ldots,k$ but $h_0\neq h_I\neq h_J$ and $\lambda_{0A}\neq\lambda_{IA}\neq\lambda_{JA}$ for $I\neq J$, and similarly for the Eq. (4.13) black holes. As we shall see, this class of black holes exhibit the same superconformal properties for small black hole separation as those of the Reissner-Nordström black hole.

VI. THE EFFECTIVE THEORY OF FIVE-DIMENSIONAL BLACK HOLES

A. Supersymmetry

It is straightforward given the moduli potential μ, Eq. (1.2), of the five-dimensional black holes to determine the metric and the torsion on the moduli space. This analysis is the same for all five-dimensional black holes, i.e., those that have the interpretation as intersecting branes in ten or eleven dimensions that preserve 1/8 of the supersymmetry and the black holes of $N=2$, $D=5$ supergravity that preserve 1/2 of the supersymmetry. So in what follows we shall not distinguish between these two cases. The moduli space is a weak HKT manifold [8] with HKT potential μ. So using [41,42], the metric and torsion are

$$ds^2=\left[\partial_{mIA}\partial_{nJB}+\sum_{s=1}^{3}(I_s)^l{}_m(I_s)^q{}_n\partial_{lIA}\partial_{qJB}\right]\mu dy^{mIA}dy^{nJB},$$

$$c=6\partial_{pIA}\partial_{qJB}\partial_{sKC}\mu\,(I_1)^p{}_m(I_2)^q{}_n(I_3)^s{}_l\times dy^{mIA}\wedge dy^{nJB}\wedge dy^{lKC}, \tag{6.1}$$

respectively, where $\{y^{mIA};\ m=1,\ldots,D-1;\ I=1,\ldots,k;\ A=1,\ldots,N\}$ label the positions of the kN black holes. The endomorphisms $\{I_r;\ r=1,2,3\}$ are associated with a constant hypercomplex structure on $\mathbb{R}^4$. These induce a hypercomplex structure on the moduli space by setting

$$(\mathbf{I}_r)^{mIA}{}_{nJB}=(I_r)^m{}_n\delta^I{}_J\delta^A{}_B, \tag{6.2}$$

which is required for the HKT structure. One can easily show that the black hole moduli space equipped with metric and torsion (6.1) and hypercomplex structure (6.2) admits an weak HKT structure.[4]

The effective theory has $N=4B$ one-dimensional supersymmetry. Both the supersymmetry multiplet and the effective action can be constructed using the general results on supersymmetric sigma models of [4,6] adapted to this case. In particular, we promote the coordinates on the moduli space to $N=4B$ superfields $y^{mIA}=y^{mIA}(t,\theta^0,\ldots,\theta^3)$ and impose the constraint

$$D_ry^{mIA}=(\mathbf{I}_r)^{mIA}{}_{nJB}D_0y^{nIA}=(I_r)^m{}_nD_0y^{nIA}, \tag{6.3}$$

where $\{D_0,\ D_r;\ r=1,2,3\}$ are the supersymmetry derivatives, i.e.,

$$D_0^2=D_r^2=i\partial_t,$$
$$D_0D_r+D_rD_0=0, \tag{6.4}$$
$$D_rD_s+D_sD_r=0,\quad r\neq s.$$

The associated $N=4B$ supersymmetric action is

$$S=-\frac{1}{2}\int dt\,d\theta^0\left[ig_{mIA,nJB}D_0y^{mIA}\partial_ty^{nJB}+\frac{1}{3!}c_{mIA,nJB,lKC}D_0y^{mIA}D_0y^{nJB}D_0y^{lKC}\right]. \tag{6.5}$$

This action describes the effective theory of five-dimensional supersymmetric black holes which preserve four supercharges.

In the special case where the positions of the harmonic functions are the same, $\mathbf{y}^A=\mathbf{y}^{IA}$, the moduli space is again a weak HKT manifold with HKT potential μ. The metric and torsion are given by

$$ds^2=\left[\partial_{mA}\partial_{nB}+\sum_{s=1}^{3}(I_s)^l{}_m(I_s)^q{}_n\partial_{lA}\partial_{qB}\right]\mu\ dy^{mA}dy^{nB},$$

$$c=6\partial_{pA}\partial_{qB}\partial_{sC}\mu(I_1)^p{}_m(I_2)^q{}_n(I_3)^s{}_ldy^{mA}\wedge dy^{nB}\wedge dy^{lC}, \tag{6.6}$$

respectively. The hypercomplex structure on the moduli space is

$$(\mathbf{I}_r)^{mA}{}_{nB}=(I_r)^m{}_n\delta^A{}_B. \tag{6.7}$$

The effective theory has again $N=4B$ supersymmetry one-dimensional supersymmetry. Using again [4,6], the $N=4B$ superfields $y^{mA}=y^{mA}(t,\theta^0,\ldots,\theta^3)$ satisfy the constraint

$$D_ry^{mA}=(\mathbf{I}_r)^{mA}{}_{nB}D_0y^{nA}=(I_r)^m{}_nD_0y^{nA}. \tag{6.8}$$

The action of the effective theory is

$$S=-\frac{1}{2}\int dt\,d\theta^0\left[ig_{mA,nB}D_0y^{mA}\partial_ty^{nB}+\frac{1}{3!}c_{mA,nB,lC}D_0y^{mA}D_0y^{nB}D_0y^{lC}\right]. \tag{6.9}$$

This completes the description of the supersymmetric effective theory actions of five-dimensional black holes.

For black holes that preserve more than four supercharges, the moduli metric is again determined by the moduli

[4] For other applications of HKT manifolds, see [45,41].

potential (1.2). However, the effective theory may contain additional fermionic multiplets. For example, for some black holes that preserve eight supercharges the moduli space admits two commuting strong HKT structures. The effective action then contains additional fermions to construct the associated multiplets. These multiplets have been described in [4,6]. Finally, we remark that the effective actions in both the above cases can also be written as a full superspace integral as

$$S=-\frac{1}{2}\int dtd^4\theta\mu. \tag{6.10}$$

B. Superconformal symmetry

In [10], it was shown that for small black hole separations the effective theory of the graviphoton electrically charged black holes of $N=2$, $D=5$ supergravity exhibits $D(2,1;0)$ superconformal symmetry. This is related to the observation that the near horizon geometry of these black holes is $AdS_2\times S^3$. However, the near horizon geometry of five-dimensional black holes that preserve 1/8 of the maximal supersymmetry and are associated with intersecting branes, and that of the black holes of the $N=2$, $D=5$ supergravity theory is also $AdS_2\times S^3$. Since this should be the case for every black hole involved in the superposition, the relevant solutions are those for which all the harmonic functions have the same positions.[5] So one expects that the effective theory (6.9) of these black holes will also exhibit $D(2,1;0)$ superconformal symmetry for small black hole separations.

The conditions for a $N=4B$ supersymmetric sigma model to exhibit superconformal symmetry have been investigated in [42] and we shall not repeat them here in detail.[6] We shall simply verify the conditions that the sigma model manifold admits a homothetic motion generated by a vector field D and that the associated one-form to D is closed. We shall then comment about the rest of the conditions.

The limit of small black hole separation is achieved by requiring that the asymptotic constants of the harmonic functions that determine the solutions vanish, i.e. $h_I\to 0$. Then following [7], we write the moduli potential as

$$\mu=\mu_1+\mu_2+\mu_3, \tag{6.11}$$

where

$$\mu_1=\int d^4x\sum_A \frac{\lambda_{1A}\lambda_{2A}\lambda_{3A}}{|\mathbf{x}-\mathbf{y}_A|^6}, \tag{6.12}$$

$$\mu_2=\int d^4x\sum_{A\neq B} \frac{\lambda_{1A}\lambda_{2A}\lambda_{3B}+\lambda_{3A}\lambda_{2A}\lambda_{1B}+\lambda_{1A}\lambda_{3A}\lambda_{2B}}{|\mathbf{x}-\mathbf{y}_A|^4|\mathbf{x}-\mathbf{y}_B|^2}, \tag{6.13}$$

$$\mu_3=\int d^4x\sum_{A\neq B\neq C} \frac{\lambda_{1A}\lambda_{2B}\lambda_{3C}}{|\mathbf{x}-\mathbf{y}_A|^2|\mathbf{x}-\mathbf{y}_A|^2|\mathbf{x}-\mathbf{y}_A|^2}. \tag{6.14}$$

There is no contribution to the moduli metric from μ_1 because it is independent from $\mathbf{y}_A$ as it can be easily seen by a change of variables in the integral. The contribution to the moduli metric due to μ_2 has a logarithmic divergence $\ln\delta/|\mathbf{y}_A-\mathbf{y}_B|^2$ for $\mathbf{x}\to\mathbf{y}_A$, where δ is a cutoff, $(|\mathbf{x}-\mathbf{y}_A|\geqslant\delta)$. However, these terms do not contribute to the moduli metric; they are eliminated passing from the potential to the moduli metric because of the differentiation. The term of μ_2 that contributes is

$$\mu_2=2\pi^2\sum_{A\neq B}(\lambda_{1A}\lambda_{2A}\lambda_{3B}+\lambda_{3A}\lambda_{2A}\lambda_{1B} +\lambda_{1A}\lambda_{3A}\lambda_{2B})\frac{\ln|\mathbf{y}_A-\mathbf{y}_B|}{|\mathbf{y}_A-\mathbf{y}_B|^2}. \tag{6.15}$$

Finally, μ_3 can be defined without regularization and it is homogeneous of degree[7] -2, i.e.,

$$y^{mA}\partial_{mA}\mu_3=-2\mu_3. \tag{6.16}$$

The homothetic motion on the moduli space is generated by the vector field

$$D^{mA}\partial_{mA}=-y^{mA}\partial_{mA}, \tag{6.17}$$

which acting on the moduli metric gives

$$\mathcal{L}_D g_{mA,nB}=2g_{mA,nB}. \tag{6.18}$$

This can be verified by an explicit calculation using the rotational invariance of the moduli potential

$$\begin{aligned} &y^{mA}(I_r)^n{}_m\partial_{nA}\mu_2=0,\\ &y^{mA}(I_r)^n{}_m\partial_{nA}\mu_3=0. \end{aligned} \tag{6.19}$$

Invariance of the effective action under special conformal transformation requires that

$$D_{mA}dy^{mA} \tag{6.20}$$

is a closed one-form, where we have used the moduli metric to lower the indices of the components of D. To show this, we first observe that the part of the metric associated with μ_3 is degenerate along D. Using this, we find that

[5]We also consider in our investigation only STU black holes.

[6]Superconformal sigma models with scalar potential have been considered in [43].

[7]In the case for which all the positions of the harmonic functions are different, μ_3 is the only contributing term. But as we shall see later, in this case the moduli metric is degenerate.

$$D_{mA}dy^{mA}=-g_{mA,nB}y^{nB}dy^{mA}$$
$$=2\pi^2 d\left[\sum_{A\neq B}\frac{\lambda_{1A}\lambda_{2A}\lambda_{3B}+\lambda_{3A}\lambda_{2A}\lambda_{1B}+\lambda_{1A}\lambda_{3A}\lambda_{2B}}{|\mathbf{y}_A-\mathbf{y}_B|^2}\right], \tag{6.21}$$

and so $D_{mA}dy^{mA}$ is closed as required. It turns out that the rest of the conditions for superconformal invariance also hold. Therefore the moduli geometry for small black hole separation exhibits a $D(2,1;0)$ superconformal invariance.

VII. THE EFFECTIVE THEORY OF FOUR-DIMENSIONAL BLACK HOLES

A. Supersymmetry

The effective theory of four-dimensional black holes which preserve 1/8 of the maximal supersymmetry is expected to have four supercharges. The dimension of the moduli space is $3nN$, where n is the number of harmonic functions of the solution and N is the number of positions of each harmonic function. The description of the effective theory is the same for all four-dimensional black holes, i.e., those that have the interpretation as intersecting branes in ten or in eleven dimensions, the black holes that are reductions of the solutions superposed with a pp wave or a KK monopole of $N=2$, $D=5$ supergravity and preserve 1/2 of the supersymmetry and the electrically charged black holes of $N=2$, $D=4$ supergravity. To keep the notation uniform, we label the positions of the former black holes as $\{\mathbf{y}^{IA};\ I=1,\dots,n;\ A=1,\dots,N\}$ and the positions of the latter black holes as $\{\mathbf{y}^{IA};\ I=0,1,\dots,k;\ A=1,\dots,N\}$ with $n=k+1$. The range of I can be different in the two cases but this would not affect our formulas below.

Next we derive the supersymmetry multiplet and the effective action of the above black holes by appropriately adapting the general results of [4] on supersymmetric one-dimensional sigma models and by comparing with the effective action of Reissner-Nordström black holes in [10]. In particular, we promote the positions of the black holes to superfields as $\mathbf{y}^{IA}=\mathbf{y}^{IA}(t,\theta^0,\dots,\theta^3)$ and add a new supersymmetry fermionic multiplet $\psi^{IA}(t,\theta^0,\dots,\theta^3)$. In addition, we impose the constraints

$$D_r y^{mIA}=\epsilon_r{}^m{}_n D_0 y^{nIA}+\delta_r^m\psi^{IA},$$
$$D_r\psi^{IA}=i\,\delta_{rn}\partial_t y^{nIA}. \tag{7.1}$$

We remark that all the four-dimensional black holes associated with intersecting branes with four harmonic functions and those that are reductions of $N=2$, $D=5$ supergravity are charged with respect one KK vector. Therefore the argument in [10] applies for the presence of the fermionic multiplets, i.e., that they are due to zero modes along the KK direction. The manifestly supersymmetric effective action of the four-dimensional black holes is

$$S=-\frac{1}{2}\int dt d^4\theta\mu(y). \tag{7.2}$$

Rewriting this action in terms of the $N=1$ superfields

$$q^{mIA}=y^{mIA}|_{\theta^r=0},\quad r=1,2,3,$$
$$\chi^{IA}=\psi^{IA}|_{\theta^r=0},\quad r=1,2,3 \tag{7.3}$$

by integrating over θ^1, θ^2, and θ^3 and by using the constraints (7.1), we find

$$S=\int dt d\theta\Big[-\frac{i}{2}g_{mIA,nJB}Dq^{mIA}\partial_t q^{nJB}-\frac{1}{2}h_{IA,JB}\chi^{IA}D\chi^{JB}$$
$$+if_{mIA,JB}\partial_t q^{mIA}\chi^{JB}$$
$$+\frac{1}{3!}c_{mIA,nJB,lKC}Dq^{mIA}Dq^{nJB}Dq^{lKC}$$
$$+\frac{1}{2}n_{mIA,nJB,KC}Dq^{mIA}Dq^{nJB}\chi^{KC}$$
$$+\frac{1}{2}m_{mIA,JB,KC}Dq^{mIA}\chi^{JB}\chi^{KC}$$
$$+\frac{1}{3!}l_{IA,JB,KC}\chi^{IA}\chi^{IB}\chi^{KC}\Big], \tag{7.4}$$

where

$$g_{mIA,nJB}=[\partial_{mIA}\partial_{nJB}+\epsilon^{lp}{}_m\epsilon_l{}^q{}_n\partial_{pIA}\partial_{qJB}]\mu,$$
$$h_{IA,JB}=\delta^{mn}\partial_{mIA}\partial_{nJB}\mu,$$
$$f_{mIA,JB}=\epsilon^{nl}{}_m\partial_{nIA}\partial_{lJB}\mu,$$
$$c_{mIA,nJB,lKC}=\frac{1}{2}\epsilon^{pqr}\epsilon_p{}^s{}_m\epsilon_q{}^t{}_n\epsilon_r{}^u{}_l\partial_{sIA}\partial_{tJB}\partial_{uKC}\mu, \tag{7.5}$$
$$n_{mIA,nJB,KC}=\left(\frac{1}{2}\epsilon^{pql}\epsilon_p{}^s{}_m\epsilon_q{}^u{}_n+\epsilon_m{}^s{}_l\delta^u{}_n\right)\partial_{sIA}\partial_{uJB}\partial_{lKC}\mu,$$
$$m_{mIA,JB,KC}=\frac{1}{2}\epsilon^{pqs}\epsilon_p{}^l{}_m\partial_{lIA}\partial_{qJB}\partial_{sKC}\mu,$$

$$l_{IA,JB,KC}=\frac{1}{2}\epsilon^{mns}\partial_{mIA}\partial_{nJB}\partial_{sKC}\mu,$$

and we have set $\theta^0=\theta$. In particular, the moduli metric is

$$ds^2=[\partial_{mIA}\partial_{nJB}+\epsilon^{lp}{}_m\epsilon_l{}^q{}_n\partial_{pIA}\partial_{qJB}]\mu dy^{mIA}dy^{nJB}. \tag{7.6}$$

In the special case where the positions of the harmonic functions are the same, $\mathbf{y}^A=\mathbf{y}^{IA}$, to construct the effective action, we again promote $\mathbf{y}^A$ to superfields and introduce N additional fermionic multiplets ψ^A. These multiplets satisfy the constraints

$$\begin{aligned}D_r y^{mA}&=\epsilon_r{}^m{}_n D_0 y^{nA}+\delta^m{}_r\psi^A,\\ D_r\psi^A&=i\,\delta_{rn}\partial_t y^{nA}.\end{aligned} \tag{7.7}$$

The effective action is again given by Eq. (7.2). Expanding the action in terms of the $N=1$ superfields

$$\begin{aligned}q^{mA}&=y^{mA}|_{\theta^r=0}, \quad r=1,2,3,\\ \chi^A&=\psi^A|_{\theta^r=0}, \quad r=1,2,3,\end{aligned} \tag{7.8}$$

we find that

$$\begin{aligned}S=\int dt\,d\theta\Big[&-\frac{i}{2}g_{mA,nB}Dq^{mA}\partial_t q^{nB}-\frac{1}{2}h_{AB}\chi^A D\chi^B\\ &+if_{mA,B}\partial_t q^{mA}\chi^B+\frac{1}{3!}c_{m,A,nB,lC}Dq^{mA}Dq^{nB}Dq^{lC}\\ &+\frac{1}{2}n_{mA,nB,C}Dq^{mA}Dq^{nB}\chi^C+\frac{1}{2}m_{mA,B,C}Dq^{mA}\chi^B\chi^C\\ &+\frac{1}{3!}l_{ABC}\chi^A\chi^B\chi^C\Big],\end{aligned} \tag{7.9}$$

where

$$\begin{aligned}g_{mA,nB}&=[\partial_{mA}\partial_{nB}+\epsilon^{lp}{}_m\epsilon_l{}^q{}_n\partial_{pA}\partial_{qB}]\mu,\\ h_{AB}&=\delta^{mn}\partial_{mA}\partial_{nB}\mu,\\ f_{mA,B}&=\epsilon^{nl}{}_m\partial_{nA}\partial_{lB}\mu,\\ c_{mA,nB,lC}&=\frac{1}{2}\epsilon^{pqr}\epsilon_p{}^s{}_m\epsilon_q{}^t{}_n\epsilon_r{}^u{}_l\partial_{sA}\partial_{tB}\partial_{uC}\mu,\\ n_{mA,nB,C}&=\left(\frac{1}{2}\epsilon^{pql}\epsilon_p{}^s{}_m\epsilon_q{}^u{}_n+\epsilon^s{}_m\delta^u{}_n\right)\partial_{sA}\partial_{uB}\partial_{lC}\mu,\\ m_{mA,B,C}&=\frac{1}{2}\epsilon^{pqs}\epsilon_p{}^l{}_m\partial_{lA}\partial_{qB}\partial_{sC}\mu,\\ l_{ABC}&=\frac{1}{2}\epsilon^{mns}\partial_{mA}\partial_{nB}\partial_{sC}\mu,\end{aligned} \tag{7.10}$$

and again we have set $\theta^0=\theta$. In particular, the moduli metric is

$$ds^2=[\partial_{mA}\partial_{nB}+\epsilon^m{}_{lp}\epsilon_l{}^q{}_n\partial_{pA}\partial_{qB}]\mu dy^{mA}dy^{nB}. \tag{7.11}$$

This completes the description of the supersymmetric effective actions for four-dimensional black holes.

For four-dimensional black holes that preserve more supersymmetry, the moduli potential is again given by Eq. (1.2). However, in the description of the effective theory one may have to add additional one-dimensional fields to describe the supersymmetry multiplets. These are required by supersymmetry as in the five-dimensional case.

B. Superconformal symmetry

The investigation of superconformal symmetry of the moduli space of four-dimensional black holes for small black hole separation is similar to that presented for the Reissner-Nordström black hole in [10]. In particular, the effective theory admits a $D(2,1;0)$ superconformal symmetry in the near horizon limit. As in the four-dimensional case, the relevant class of black holes are those that exhibit regular near horizon geometry. These are the black holes that have four harmonic functions[8] with the same centers for which the near horizon geometry at every center is $AdS_2\times S^2$.

The conditions for a sigma model action such as Eq. (7.9) to exhibit superconformal symmetry have been given in [10] and we shall not repeat them here. In what follows, we shall show that our moduli metric admits a homothetic motion generated by a vector field D and that the associated one-form of D is closed. To begin, we write

$$g_{mA,nB}=G^{kl}_{mn}\partial_{kA}\partial_{lB}\mu, \tag{7.12}$$

where

$$G^{kl}_{mn}=\delta^k_m\delta^l_n+\epsilon^{rk}{}_m\epsilon_r{}^l{}_n. \tag{7.13}$$

Since we expect a close relationship between the superconformal properties of the five-dimensional black holes and those of the four-dimensional ones, we take the vector field D which generates the homothetic motion to be the following:

$$D^{mA}\partial_{mA}=\frac{2}{h}y^{mA}\partial_{mA}, \tag{7.14}$$

where h is a constant which will be determined. Using our ansatz for D and the expression (7.12), we find that D is a homothety if

$$(y^{mA}\partial_{mA}-h)\mu=\frac{h}{2}K, \tag{7.15}$$

where K is in the kernel of the operator

[8]We also consider only the black holes and string solutions of the $N=2$, $D=5$ supergravity associated with the STU model.

$$G_{ABmn}=G^{kl}_{mn}\partial_{kA}\partial_{lB}, \tag{7.16}$$

which is used to find the metric on the moduli space from the moduli potential.

As in the case of five-dimensional black holes above, we write the moduli potential as

$$\mu=\mu_1+\mu_2+\mu', \tag{7.17}$$

where

$$\mu_1=\int d^3x\sum_A \frac{\lambda_{1A}\lambda_{2A}\lambda_{3A}\lambda_{4A}}{|\mathbf{x}-\mathbf{y}_A|^4}, \tag{7.18}$$

$$\mu_2=\int d^3x\sum_{A\neq B}\left[\frac{\lambda_{1A}\lambda_{2A}\lambda_{3A}\lambda_{4B}}{|\mathbf{x}-\mathbf{y}_A|^3|\mathbf{x}-\mathbf{y}_B|}+\text{cyclic in}(1,2,3,4)\right], \tag{7.19}$$

and μ' contains the rest of the terms. Both μ_1 and μ_2 contain divergent terms which however do not contribute to the moduli metric. In particular, μ_1 is independent form the positions of the black holes and so it does not contribute. Putting a cut off $|\mathbf{x}-\mathbf{y}_A|\geqslant\delta$, we can evaluate μ_2 to find

$$\mu_2=4\pi\sum_{A\neq B}\left[\lambda_{1A}\lambda_{2A}\lambda_{3A}\lambda_{4B}\frac{\ln|\mathbf{y}_A-\mathbf{y}_B|+(1-\ln\delta)}{|\mathbf{y}_A-\mathbf{y}_B|}+\text{cyclic in}(1,2,3,4)\right]. \tag{7.20}$$

Next using

$$G_{CDmn}\frac{1}{|\mathbf{y}_A-\mathbf{y}_B|}=0, \tag{7.21}$$

we see that the divergent part does not contribute. So ignoring the divergent part, we see that

$$(y^{mA}\partial_{mA}+1)\mu_2=-\frac{1}{2}K, \tag{7.22}$$

where

$$K=\frac{\pi}{4}\sum_{A\neq B}\left[\lambda_{1A}\lambda_{2A}\lambda_{3A}\lambda_{4B}\frac{1}{|\mathbf{y}_A-\mathbf{y}_B|}+\text{cyclic in}(1,2,3,4)\right], \tag{7.23}$$

which is in the kernel of G_{CDmn}. Next, we can observe that μ' is homogeneous of degree -1 and so

$$(y^{mA}\partial_{mA}+1)\mu'=0. \tag{7.24}$$

From all these, we find that Eq. (7.15) holds for $h=-1$ and

$$D=-2y^{mA}\partial_{mA} \tag{7.25}$$

is a homothetic vector field. To generate special conformal transformations, the associated form of D should be closed as in the case of five-dimensional black holes. In particular, one can show that

$$D_{mA}=-2g_{mA,nB}y^{nB}=\partial_{mA}K. \tag{7.26}$$

For this, we have used the rotational invariance of μ under $SO(3)$, i.e.,

$$y^{mA}\epsilon^{nl}{}_m\partial_{lB}\mu=0. \tag{7.27}$$

The above properties of the moduli metric indicate that at small black hole separation the effective theory admits an $SL(2,\mathbb{R})$ symmetry. It turns out that the rest of the conditions for superconformal invariance of [10] can also be verified. So the effective action admits a $D(2,1;0)$ superconformal symmetry.

VIII. BLACK HOLES AND SPECIAL KÄHLER GEOMETRY

A. The black hole solutions

The action of $N=2$ four-dimensional supergravity with $n+1$ vectors $F^I=dA^I$ and n complex scalars z^a has been found in [44]. A class of such systems can be described in terms of a holomorphic homogeneous of degree two potential $F=F(X^I)$. It follows that

$$\begin{aligned} F&=\frac{1}{2}F_IX^I,\\ F_I&=F_{IJ}X^J,\\ X^IF_{IJK}&=0,\\ X^IF_{IJKL}&=-F_{JKL}, \end{aligned} \tag{8.1}$$

where $F_I=(\partial/\partial X^I)F$, $I=0,\ldots,n$, and similarly for higher order derivatives. Next we set

$$\begin{aligned} e^{-K}&\equiv i(\overline{X^I}F_I-X^I\overline{F_I}),\\ N_{IJ}&=i(\overline{F}_{IJ}-F_{IJ}),\\ \mathcal{N}_{IJ}&=\overline{F}_{IJ}+\frac{i}{(XNX)}(NX)_I(NX)_J, \end{aligned} \tag{8.2}$$

where $(NX)_I=N_{IJ}X^J$, $(N\overline{X})_I=N_{IJ}\overline{X}^J$, $XNX=X^IX^JN_{IJ}$ and $\overline{X}N\overline{X}=\overline{XNX}$. The existence of such a potential F is not always guaranteed. The field equations of $N=2$ four-dimensional supergravity are invariant under symplectic reparametrizations. It has been shown that one may use this to pass from a solution which possesses a potential to one which does not. However, throughout this section we shall limit ourselves to configurations which possess a potential F. The coordinates X^I are holomorphic functions of z^a. The bosonic part of the $N=2$ four dimensional supergravity action is

$$S=\int d^4x\sqrt{|g|}(R+2[e^K N_{IJ}+e^{2K}(N\overline{X})_I(NX)_J]\partial_\mu X^I\partial^\mu\overline{X}^J$$
$$+i(\mathcal{N}_{IJ}-\overline{\mathcal{N}}_{IJ})F^I{}_{\mu\nu}F^{J\mu\nu})$$
$$-\frac{1}{2}(\mathcal{N}_{IJ}+\overline{\mathcal{N}}_{IJ})\epsilon^{\mu\nu\rho\sigma}F^I{}_{\mu\nu}F^J{}_{\rho\sigma}, \tag{8.3}$$

where we have chosen $\epsilon^{0123}=+1$. The field equations of Eq. (8.3) are as follows. The Einstein field equation is

$$\sqrt{|g|}(G_{\mu\nu}+2[e^K N_{IJ}+e^{2K}(N\overline{X})_I(NX)_J]\partial_{(\mu}X^I\partial_{\nu)}\overline{X}^J$$
$$+2i(\mathcal{N}_{IJ}-\overline{\mathcal{N}}_{IJ})F^I_{\mu\lambda}F^{J\lambda}_{\nu})-\frac{1}{2}\sqrt{|g|}g_{\mu\nu}(2[e^K N_{IJ}$$
$$+e^{2K}(N\overline{X})_I(NX)_J]\partial_\lambda X^I\partial^\lambda\overline{X}^J$$
$$+i(\mathcal{N}_{IJ}-\overline{\mathcal{N}}_{IJ})F^I_{\rho\sigma}F^{J\rho\sigma})=0, \tag{8.4}$$

the vector potential field equations are

$$8\partial_\mu(\sqrt{|g|}[\mathrm{Im}\,\mathcal{N}_{IJ}F^{J\mu\nu}+\mathrm{Re}\,\mathcal{N}_{IJ}{}^\star F^{J\mu\nu}])=0, \tag{8.5}$$

and the field equations of the scalars z^a and $\overline{z^a}$ are

$$\Big\{-2\partial_\mu(\sqrt{|g|}[e^K N_{LJ}+e^{2K}(N\overline{X})_L(NX)_J]\partial^\mu\overline{X}^J)$$
$$+2\sqrt{|g|}\partial_L(e^K N_{IJ}+e^{2K}(N\overline{X})_I(NX)_J)\partial_\mu X^I\partial^\mu\overline{X}^J$$
$$+i\sqrt{|g|}\partial_L(\mathcal{N}_{IJ}-\overline{\mathcal{N}}_{IJ})F^I{}_{\mu\mu}F^{J\mu\nu}-\frac{1}{2}\partial_L(\mathcal{N}_{IJ}$$
$$+\overline{\mathcal{N}}_{IJ})\epsilon^{\mu\nu\rho\sigma}F^I{}_{\mu\nu}F^J{}_{\rho\sigma}\Big\}\partial_a X^L=0 \tag{8.6}$$

and

$$\Big\{-2\partial_\mu(\sqrt{|g|}[e^K N_{LJ}+e^{2K}(N\overline{X})_J(NX)_L]\partial^\mu X^J)$$
$$+2\sqrt{|g|}\partial_{\overline{L}}(e^K N_{IJ}+e^{2K}(N\overline{X})_I(NX)_J)\partial_\mu X^I\partial^\mu\overline{X}^J$$
$$+i\sqrt{|g|}\partial_{\overline{L}}(\mathcal{N}_{IJ}-\overline{\mathcal{N}}_{IJ})F^I{}_{\mu\mu}F^{J\mu\nu}-\frac{1}{2}\partial_{\overline{L}}(\mathcal{N}_{IJ}$$
$$+\overline{\mathcal{N}}_{IJ})\epsilon^{\mu\nu\rho\sigma}F^I{}_{\mu\nu}F^J{}_{\rho\sigma}\Big\}\partial_{\overline{a}}\overline{X}^L=0, \tag{8.7}$$

respectively. In the equations above, ∂_a and $\partial_{\overline{a}}$ denote the partial derivatives with respect to z^a and $\overline{z^a}$, respectively.

The static black hole solution which we shall consider has X^I real. In which case, F and all its derivatives can be chosen to be purely imaginary. The solution is

$$ds^2=-e^K dt^2+e^{-K}d\mathbf{x}^2,$$
$$A^I{}_0=e^K X^I,$$
$$A^I{}_m=0, \tag{8.8}$$
$$2iF_I=H_I,$$

where

$$H_I=h_I+\sum_A\frac{\lambda_{IA}}{|\mathbf{x}-\mathbf{y}_{IA}|} \tag{8.9}$$

are harmonic functions. Using the definition of K, we find that for this solution $e^{-K}=4iF$. Now the last equation in Eq. (8.8) can be thought of as expressing $n+1$ real scalars in terms of $n+1$ real harmonic functions. From these, n scalars are associated with the scalar fields of the supergravity theory and one with the components of the metric as in the five-dimensional case explained in section four. The centers of the harmonic functions $\{\mathbf{y}_{IA};\ I=0,\dots,n;\ A=1,\dots,n_J\}$ determine the positions of the black holes and $\{\lambda_{IA};\ I=0,\dots,n;\ A=1,\dots,n_I\}$ are their electric charges. The above solution has delta function sources in the coordinate system that we are using to describe it. The appropriate source terms which should be added to the supergravity action are

$$S_{\mathrm{source}}=\sum_{I,A}\int d\tau_{IA}\Big(8\pi e^{(1/2)K}(X^I+\overline{X}^I)\lambda_{IA}$$
$$-16\pi A^I_\mu\frac{dy_{IA}{}^\mu}{d\tau_{IA}}\lambda_{IA}\Big), \tag{8.10}$$

where τ_{IA} is the proper time associated with the centres $\mathbf{y}_{IA}$ defined with respect to the metric g.

B. Perturbations

In order to determine the low energy behavior of these solutions we allow the centres $\mathbf{y}_{IA}$ to depend on t. We also make the following additional first order in the velocities perturbations to the fields:

$$ds^2\to ds^2+2e^K p_n dt\,dx^n,$$
$$A^I{}_0 dt\to A^I{}_0 dt+(D^I{}_n-e^K X^I p_n)dx^n, \tag{8.11}$$
$$X^I\to X^I+iY^I,$$

where Y^I is real, and p_n, $D^I{}_n$ $(n=1,2,3)$ and Y^I are to be determined by solving the equations of motion up to first order in the velocities.[9] It turns out that using this perturbation ansatz only the n-component of the gauge equation and the $0n$ component of the Einstein equation together with the field equations of the scalar z^a are modified by terms first order in the velocities. As the scalar perturbation is imaginary, the conjugate scalar equation does not contain any additional information.

In particular the perturbed Einstein equation including the sources from Eq. (8.10) gives

[9]One can perturb all of the fields in the theory around a solution, but in this case the perturbation we have considered will suffice.

$$-\frac{1}{2}e^K\partial^l(\partial_n p_l-\partial_l p_n)$$
$$+2X^I\partial_0\partial_n H_I+4ie^K\partial^l F_I(\partial_n D^I{}_l-\partial_l D^I{}_n)$$
$$=8\pi X^I\sum \lambda_{IA}\delta(\mathbf{x}-\mathbf{y}_{IA})v_{IAn} \tag{8.12}$$

and the perturbed gauge equation including the sources gives

$$4\partial_0\partial^n H_I+8i\partial_m(e^K(F_{IJ}-F^{-1}F_IF_J)(\partial^m D^{In}-\partial^n D^{Im}))$$
$$+8i\partial_m(e^{2K}F_I(\partial^m p^n-\partial^n p^m))$$
$$-2\epsilon^{mml}\partial_m Y^L\partial_l(F^{-1}F_{IL})-2\epsilon^{mnl}\partial_m(F^{-2}Y^LF_I)\partial_l F_L$$
$$=16\pi\sum \lambda_{IA}\delta(\mathbf{x}-\mathbf{y}_{IA})v^n_{IA}. \tag{8.13}$$

To proceed we contract Eq. (8.13) with $\frac{1}{2}X^I$ and subtract it from Eq. (8.12). This leads to the simpler expression

$$\partial^m\left[\frac{3i}{8}F^{-2}(\partial_m p_n-\partial_n p_m)+F^{-2}F_I(\partial_m D^I{}_n-\partial_n D^I{}_m)\right.$$
$$\left.+2F^{-1/2}F_J\epsilon^l{}_{nm}\partial_l(F^{-3/2}Y^J)\right]=0. \tag{8.14}$$

The perturbed scalar equation including sources gives

$$\partial_a X^L\left\{\partial_m\left(\left(-iF^{-1}F_{LJ}+\frac{1}{2}F^{-2}F_LF_J\right)\partial^m Y^J\right)\right.$$
$$+\frac{i}{2}F^{-2}\partial^m Y^J(F_J\partial_m F_L-F_L\partial_m F_J)-\frac{3i}{2}F^{-2}\partial^m F_J\partial_m F_L$$
$$-\frac{1}{2}Y^J\partial_m(F^{-2}F_L\partial^m F_J)-\frac{i}{2}\epsilon^{mnr}(\partial_m(F^{-1}F_{JL})$$
$$+F^{-2}F_J\partial_m F_L)(\partial_n D^J{}_r-\partial_r D^J{}_n)$$
$$\left.+\frac{1}{4}F^{-3/2}\epsilon^{mmr}\partial_m(F^{-1/2}F_L)(\partial_n p_r-\partial_r p_n)\right\}=0. \tag{8.15}$$

We shall not continue to present the solutions to these second order (with respect to spatial derivatives) equations in this section, instead we shall first evaluate the term in the action which is quadratic in the velocities. From there it will become clear that the perturbations solve a set of first order equations.

C. The moduli metric

To compute the moduli metric, we must substitute the solution to the perturbed field equations found in the previous section into the total action (including source terms) and compute the part which is second order in velocities. It is expected, as a consequence of the BPS condition, that the zeroth and first order contributions to the action vanish. Substituting the perturbation ansatz into the action (including the sources), and collecting the terms quadratic in the velocities, we find

$$S^{(2)}=\int d^4x\left[-16\partial_0F_I\partial_0(FX^I)-8\pi\sum e^{-K}X^I\lambda_{IA}|\mathbf{v}_{IA}|^2\delta(\mathbf{x}-\mathbf{y}_{IA})-\frac{1}{4}e^{2K}(\partial_m p_n-\partial_n p_m)(\partial^m p^n-\partial^n p^m)\right.$$
$$-2(\partial^m D^{In}-\partial^n D^{Im})(\partial_m k_{In}-\partial_n k_{Im})+2ie^K(-F_{IJ}+F^{-1}F_IF_J)(\partial_m D^I{}_n-\partial_n D^I{}_m-e^KX^I(\partial_m p_n-\partial_n p_m))$$
$$\times(\partial^m D^{Jn}-\partial^n D^{Jm}-e^KX^J(\partial^m p^n-\partial^n p^m))+\left(\frac{1}{2}F^{-2}F_IF_J-F^{-1}F_{IJ}\right)\partial_m Y^I\partial^m Y^J-F^{-2}Y^I\partial^m Y^JF_J\partial_m F_I$$
$$+\frac{3}{2}F^{-2}Y^KY^L\partial_m F_K\partial^m F_L+Y^L\epsilon^{mnr}(\partial_m(F^{-1}F_{JL})+F^{-2}F_J\partial_m F_{\mathbf{L}})(\partial_n D^J{}_r-\partial_r D^J{}_n)$$
$$\left.+\frac{i}{2}F^{-3/2}Y^L\epsilon^{mnr}\partial_m(F^{-1/2}F_L)(\partial_n p_r-\partial_r p_n)\right], \tag{8.16}$$

where

$$k_I{}^n=\sum \frac{\lambda_{IA}v_{IA}{}^n}{|\mathbf{x}-\mathbf{y}_{IA}|}. \tag{8.17}$$

It should be noted that on varying the fields p_n, $D^I{}_n$ and Y^I one obtains the first order in the velocities field equations of the previous section. To simplify the action we set

$$Q_{mn}=\partial_m p_n-\partial_n p_m-\frac{4}{3}H_I(\partial_m D^I{}_n-\partial_n D^I{}_m)$$
$$+\frac{4i}{3}F^{3/2}F_L\epsilon_{mn}{}^r\partial_r(F^{-3/2}Y^L)$$
$$-4iF^{1/2}Y^L\epsilon_{mn}{}^r\partial_r(F^{-1/2}F_L), \tag{8.18}$$

$$Q^I{}_{mn}=\partial_m D^I{}_n-\partial_n D^I{}_m+2B^{IJ}(\partial_m k_{Jn}-\partial_n k_{Jm})+\epsilon_{mn}{}^l\partial_l Y^I,$$

where

$$B_{IJ}=F^{-1}(F_{IJ}-F^{-1}F_IF_J); \tag{8.19}$$

B^{IJ} is the inverse of the matrix B_{IJ}. We shall assume that this inverse exists, this is certainly true in many interesting cases, like those of intersecting D-branes. Remarkably then, the second order action simplifies to

$$S^{(2)}=\int d^4x\bigg[-16\partial_0F_I\partial_0(FX^I)-8\pi\sum e^{-K}X^I\lambda_{IA}|\mathbf{v}_{IA}|^2\times\delta(\mathbf{x}-\mathbf{y}_{IA})+2B^{IJ}(\partial_m k_{In}-\partial_n k_{Im})(\partial^m k_J{}^n-\partial^n k_J{}^m)+\frac{3}{4}e^{2K}Q_{mn}Q^{mn}-\frac{1}{2}B_{IJ}Q^I{}_{mn}Q^{Jmn}\bigg]. \tag{8.20}$$

To proceed, we shall take $Q=0$ and $Q^I=0$. It turns out that these conditions are sufficient for solving the perturbed field equations. Furthermore, the portion of the above action which is independent of Q and Q^I is precisely that which leads to the effective action for the black holes which possesses the expected $N=4$ worldline supersymmetry. Solving $Q=Q^I=0$, we find

$$Y^I=\frac{1}{4\pi}\int d^3\mathbf{z}\frac{1}{|\mathbf{x}-\mathbf{z}|}\epsilon^{lmn}\partial_l(B^{IJ}(\partial_m k_{Jn}-\partial_n k_{Jm}))(\mathbf{z}),$$

$$D^I_n=\frac{1}{2\pi}\int d^3\mathbf{z}\frac{1}{|\mathbf{x}-\mathbf{z}|}\partial^l(B^{IJ}(\partial_l k_{Jn}-\partial_n k_{Jl}))(\mathbf{z}),$$

$$p_n=\frac{1}{4\pi}\int d^3\mathbf{z}\frac{1}{|\mathbf{x}-\mathbf{z}|}\partial^l(16iFX^I(\partial_l k_{In}-\partial_n k_{Il})+4i\epsilon_{ln}{}^r(F_L\partial_rY^L-Y^L\partial_rF_L))(\mathbf{z}). \tag{8.21}$$

Substituting the solutions for the perturbations back into $S^{(2)}$, we find that

$$S^{(2)}=\int d^4x\bigg[-16\partial_0F_I\partial_0(FX^I)-8\pi\sum e^{-K}X^I\lambda_{IA}|\mathbf{v}_{IA}|^2\times\delta(\mathbf{x}-\mathbf{y}_{IA})+2B^{IJ}(\partial_m k_{In}-\partial_n k_{Im})(\partial^m k_J{}^n-\partial^n k_J{}^m)\bigg]. \tag{8.22}$$

The moduli metric can be read from $S^{(2)}$. To analyze the geometry of the moduli space we use the identities

$$B_{IJ}X^J=\frac{1}{3}F^{-1}F_I,\qquad B_{IJ}\partial_\mu(FX^J)=\partial_\mu F_I. \tag{8.23}$$

It then follows that

$$\partial_{mIA}F=\frac{i}{2}X^I\partial_m\left[\frac{\lambda_{IA}}{|\mathbf{x}-\mathbf{y}_{IA}|}\right] \tag{8.24}$$

and

$$\partial_{mIA}\partial_{nJB}F^2=-\frac{1}{4}e^{-K}X^I\delta_{IJ}\delta_{AB}\partial_m\partial_n\left[\frac{\lambda_{IA}}{|\mathbf{x}-\mathbf{y}_{IA}|}\right]-\frac{1}{2}B^{IJ}\partial_m\left[\frac{\lambda_{IA}}{|\mathbf{x}-\mathbf{y}_{IA}|}\right]\partial_n\left[\frac{\lambda_{JB}}{|\mathbf{x}-\mathbf{y}_{JB}|}\right], \tag{8.25}$$

where there is no sum over I or J and ∂_{mIA} denotes partial differentiation with respect to y_{mIA}. Using all the above we find

$$S^{(2)}=\int dt\frac{1}{2}g_{mIA\,nJB}v_{IA}{}^m v_{JB}{}^n, \tag{8.26}$$

where

$$g_{mIA,nJB}=\partial_{mIA}\partial_{nJB}\mu-\partial_{nIA}\partial_{mJB}\mu+\delta_{mn}\delta^{rl}\partial_{rIA}\partial_{lJB}\mu \tag{8.27}$$

and

$$\mu=-16\int d^3xF^2. \tag{8.28}$$

So the moduli metric is

$$ds^2=g_{mIA,nJB}dy^{mIA}dy^{nJB}. \tag{8.29}$$

As an example we may consider the STU model with $n=4$. For this case the potential function F is given by

$$F(X^I)=i(X^0X^1X^2X^3)^{1/2}. \tag{8.30}$$

This leads to the moduli potential

$$\mu=16\int d^3xH_0H_1H_2H_3. \tag{8.31}$$

From this we observe that the moduli space metric possesses the expected $N=4$ supersymmetry together with the appropriate superconformal symmetry for those black holes associated with harmonic functions with the same centers. The moduli space metric is generated by a potential function in agreement with the conjecture (1.2).

IX. CONCLUDING REMARKS

We have proposed that the moduli metric of a large class of four- and five-dimensional black holes can be determined by a moduli potential. In turn this can be determined by the components of the metric of the black hole solution as in Eq. (1.2). Then we have provided evidence that such choice gives consistent results. In particular, it describes all the black hole moduli metrics that have been computed explicitly. In some cases, one can argue under certain assumptions that the expression for the moduli potential (1.2) can be shown using duality. Moreover, the associated effective theories of black holes which are constructed using Eq. (1.2) exhibit the expected superconformal behavior at small black hole separations.

One can extend our construction to find a U-duality invariant expression for the moduli potential. However, for this to be consistent one should use U-duality invariant black hole solutions [46]; for recent work see for example [47]. In four dimensions, such black holes carry the charges of all branes.[10] It would be interesting to see whether our moduli potential formula still applies in this case.

It is clear from our results that the effective theories of black holes that have regular horizons exhibit superconformal symmetry at small black hole separations. This can prove useful in understanding multi-black hole quantum mechanics using the suggestion of [50] adapted for black holes [51,52] and black hole moduli spaces [7,42,10].

[10]Note however that in string theory [48] as well as in supergravity [49], D6-branes repel D0-branes; although this may change if other brane charges are present.

ACKNOWLEDGMENTS

We thank A. Strominger for helpful suggestions and comments. G.P. also thanks G. W. Gibbons for many discussions on four-dimensional black hole moduli spaces. J.G. thanks EPSRC for financial support. G.P. is supported by the Royal Society. This work is partly supported by the PPARC grant PPA/G/S/1998/00613.

[1] G. W. Gibbons and P. J. Ruback, Phys. Rev. Lett. **57**, 1492 (1986).
[2] R. C. Ferrell and D. M. Eardley, Phys. Rev. Lett. **59**, 1617 (1987).
[3] K. Shiraishi, Nucl. Phys. **B402**, 399 (1993).
[4] R. A. Coles and G. Papadopoulos, Class. Quantum Grav. **7**, 427 (1990).
[5] C. M. Hull, ''The geometry of supersymmetric quantum mechanics,'' hep-th/9910028.
[6] G. W. Gibbons, G. Papadopoulos, and K. Stelle, Nucl. Phys. **B508**, 623 (1997).
[7] J. Michelson and A. Strominger, J. High Energy Phys. **09**, 005 (1999); R. Britto-Pacumio, J. Michelson, A. Strominger, and A. Volovich, ''Lectures on superconformal quantum mechanics and multi-black hole moduli spaces,'' hep-th/9911066.
[8] P. S. Howe and G. Papadopoulos, Phys. Lett. B **379**, 80 (1996).
[9] J. Gutowski and G. Papadopoulos, Phys. Lett. B **472**, 45 (2000).
[10] A. Maloney, M. Spradlin, and A. Strominger, ''Superconformal Multi-Black Hole Moduli Spaces in Four Dimensions,'' hep-th/9911001.
[11] K. Behrndt, D. Lust, and W. A. Sabra, Nucl. Phys. **B510**, 264 (1998).
[12] G. Papadopoulos and P. K. Townsend, Phys. Lett. B **380**, 273 (1996).
[13] I. R. Klebanov and A. A. Tseytlin, Nucl. Phys. **B475**, 179 (1996).
[14] A. Tseytlin, Nucl. Phys. B (Proc. Suppl.) **55B**, 219 (1997).
[15] V. Balasubramanian, F. Larsen, and R. G. Leigh, Phys. Rev. D **57**, 3509 (1998).
[16] A. Strominger and C. Vafa, Phys. Lett. B **379**, 99 (1996).
[17] C. G. Callan, Jr. and J. M. Maldacena, Nucl. Phys. **B472**, 591 (1996).
[18] P. K. Townsend, Phys. Lett. B **350**, 184 (1995).
[19] M. J. Duff and K. S. Stelle, Phys. Lett. B **253**, 113 (1991).
[20] R. Guyen, Phys. Lett. B **276**, 49 (1992).
[21] G. T. Horowitz and A. Strominger, Nucl. Phys. **B360**, 197 (1991).
[22] M. J. Duff and J. X. Lu, Phys. Lett. B **273**, 409 (1991).
[23] A. Dabholkar, G. W. Gibbons, J. A. Harvey, and F. Ruiz-Ruiz, Nucl. Phys. **B340**, 33 (1990).
[24] C. G. Callan, Jr., J. A. Harvey, and A. Strominger, Nucl. Phys. **B367**, 60 (1991).
[25] A. A. Tseytlin, Nucl. Phys. **B475**, 149 (1996).
[26] C. V. Johnson, A. W. Peet, and J. Polchinski, Phys. Rev. D **61**, 086001 (2000).
[27] M. Günaydin, G. Sierra, and P. K. Townsend, Phys. Lett. **133B**, 72 (1983); Nucl. Phys. **B242**, 244 (1984).
[28] B. de Wit and A. van Proeyen, Phys. Lett. B **293**, 95 (1992).
[29] W. A. Sabra, Mod. Phys. Lett. A **13**, 239 (1998); A. Chamseddine and W. A. Sabra, Phys. Lett. B **426**, 36 (1998).
[30] A. Chamseddine and W. A. Sabra, Phys. Lett. B **460**, 63 (1999).
[31] M. Cvetic and D. Youm, Phys. Rev. D **53**, 584 (1996).
[32] M. Cvetic and A. Tseytlin, Phys. Lett. B **366**, 95 (1996); Phys. Rev. D **53**, 5619 (1996); **55**, 3907(E) (1997).
[33] A. Chou, R. Kallosh, J. Rahmfeld, S-J. Ray, M. Shmakova, and W. K. Wong, Nucl. Phys. **B508**, 147 (1997).
[34] A. H. Chamseddine, S. Ferrara, G. W. Gibbons, and R. Kallosh, Phys. Rev. D **55**, 3647 (1997).
[35] G. W. Gibbons, Nucl. Phys. **B207**, 337 (1982).
[36] R. Kallosh and A. Peet, Phys. Rev. D **46**, 5223 (1992).
[37] G. W. Gibbons and P. K. Townsend, Phys. Rev. Lett. **71**, 3754 (1993).
[38] I. Antoniadis, S. Ferrara, and T. R. Taylor, Nucl. Phys. **B460**, 489 (1996).
[39] I. Gauda, S. Mahapatra, T. Mohaupt, and W. A. Sabra, Class. Quantum Grav. **16**, 419 (1999).
[40] R. Kallosh, A. Linde, and M. Shmakova, ''Supersymmetric Multiple Basin Attractors,'' hep-th/9910021.
[41] G. Grantcharov and Y. S. Poon, ''Geometry of hyper-Kähler Connections with Torsion,'' math.dg/9908015.
[42] J. Michelson and A. Strominger, ''The geometry of (Super)-conformal Quantum Mechanics,'' HUTP-99/A045, hep-th/9907191.
[43] G. Papadopoulos, ''Conformal and Superconformal Mechanics,'' hep-th/0002007.
[44] B. de Wit and A. van Proeyen, Nucl. Phys. **B245**, 89 (1984); E. Cremmer, C. Kounnas, A. van Proeyen, J. P. Derendinger,

S. Ferrara, B. de Wit, and L. Girardello, *ibid.* **B250**, 385 (1985).
[45] G. Papadopoulos and A. Teschendorff, Phys. Lett. B **443**, 159 (1998); ''Grassmannians, Calibrations and Five-Brane Intersections,'' hep-th/9811034.
[46] M. Cvetic and C. M. Hull, Nucl. Phys. **B480**, 296 (1996).
[47] M. Bertolini, P. Fre, and M. Trigiante, Class. Quantum Grav. **16**, 2987 (1999); M. Bertolini and M. Trigiante, ''Regular R-R and NS-NS BPS Black Holes,'' hep-th/9910237; ''Regular BPS Black Holes: Macroscopic and Microscopic Description of the Generating solution,'' hep-th/0002191.
[48] J. Polchinski, ''Tasi lectures on D-branes,'' hep-th/9611050.
[49] M. S. Costa and G. Papadopoulos, Nucl. Phys. **B510**, 217 (1998).
[50] V. de Alfaro, S. Fubini, and G. Furlan, Nuovo Cimento A **34**, 569 (1976).
[51] P. Claus, M. Derix, R. Kallosh, J. Kumar, P. K. Townsend, and A. van Proeyen, Phys. Rev. Lett. **81**, 4553 (1998).
[52] G. W. Gibbons and P. K. Townsend, Phys. Lett. B **454**, 187 (1999).